Band-Limited Interpolation from Unevenly Spaced Sampled Data

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Abstract—This paper addresses the problem of reconstructing a band-limited signal from a finite number of its unevenly spaced sampled data. A Fourier analysis of the available unevenly spaced sampled data is presented. The result is utilized to develop an interpolation scheme from the available data. Conditions for accurate reconstruction are examined. Algorithms to implement the reconstruction scheme are discussed. The method's application in one-dimensional and two-dimensional reconstruction problems are shown.

I. INTRODUCTION

This paper considers the problem of reconstructing a space-limited multidimensional function when the available data are presented on unevenly spaced sampled points on the frequency plane of the function. The sampled points are assumed to result from a known or unknown transformation of Cartesian coordinates on the frequency plane. This type of data is often found in inverse problems. In this case, the available data are a consequence of multiple measurements in a lower dimension when certain parameters of the inverse (imaging) system are varied. The most common examples are found in computer assisted tomography [1], [2]. The parameters that are varied are usually the angle of propagation and/or the frequency of excitation. These parameters, along with the physical model of the imaging system, determine the positions of the unevenly spaced sampled points on the Cartesian frequency plane.

Recovery of band-limited signals\footnote{In our study, the band-limited signal is the spatial frequency domain function; i.e., its inverse spatial Fourier transform is space-limited.} from their unevenly spaced data has been examined by many researchers. There are various techniques to reconstruct a space-limited multidimensional function from its unevenly spaced frequency domain data (e.g., [3]–[12]). One of the proposed schemes is the Unified Fourier Reconstruction (UFR) method [13], [14]. UFR is an interpolation scheme to reconstruct the Fourier (frequency plane) transform of the function. The resultant data are then used in inverse discrete Fourier transform routines to reconstruct the desired function.

In this paper, we approach the problem of reconstruction from unevenly spaced sampled data by analyzing the spectrum [inverse spatial Fourier transform (IFT)] of the available data (Section III-A and B). We discuss conditions for accurate reconstruction as the parameters of the imaging system are varied (Section III-C). The results are then utilized to develop an accurate reconstruction scheme (Section IV). We show that the UFR method is a computationally efficient way of implementing the reconstruction scheme. We also present a scheme to implement the reconstruction method when the transformation that resulted in the unevenly spaced data on the frequency plane is unknown (Section V). This scheme, unlike the Generalized Hexagonal Tessellation method suggested by Clark et al. [12] does not require exhaustive search and yields a unique solution for the multidimensional problems. Our analysis is based on examination of one-dimensional inverse systems. The study of reconstruction in multidimensional inverse systems is similar in most respects to the treatment of the one-dimensional case. We will bring out and treat the problems associated with multidimensional cases that do not arise in the one-dimensional systems. Examples will be provided on specific one-dimensional and two-dimensional inverse systems.

II. BACKGROUND

Consider a one-dimensional space-limited function $f(x)$. We assume that $f(x)$ is centered at $x = 0$, and $f(x) = 0$ for $|x| > X_0$, where $X_0$ is a known constant. We denote the spatial Fourier transform (FT) of $f(x)$ by $F(\mu)$. Let $T(\cdot)$ be a differentiable and one-to-one transformation. We define

$$\alpha = T(\mu). \tag{1}$$

We denote the inverse of the $T$ transformation by $S$. Thus,

$$\mu = S(\alpha). \tag{2}$$

Note that the Jacobian of the transformation from $\alpha$ to $\mu$ is equal to the inverse of the Jacobian of the transformation from $\mu$ to $\alpha$; i.e.,

$$\left| \frac{dT}{d\mu} \right| = \left| T'(\mu) \right| = \frac{1}{\left| S'(\alpha) \right|} = \frac{1}{\left| \frac{ds}{d\alpha} \right|}. \tag{3}$$

We assume that $\left| T'(\mu) \right| \neq 0$ or $\infty$ for all values of $\mu$; we will examine the implications of $\left| T'(\mu) \right| = 0$ or $\infty$ in the later sections.

We define the mapping of $F(\cdot)$ into the $\alpha$-domain by

$$G(\alpha) \equiv F[S(\alpha)] = F(\mu). \tag{4}$$
We assume that \( G(\alpha) \) is known at a finite number of equally spaced points in the \( \alpha \)-domain, e.g., \( \alpha_n = n\Delta \), for \( n = 0, \pm 1, \pm 2, \cdots, \pm K \). Our purpose is to reconstruct \( f(x) \) or \( F(\mu) \) from the available set of data, i.e., \( G(\alpha_n) \).

A. Shannon Reconstruction [5]

If \( G(\alpha) \) were a band-limited function, one could utilize Shannon's sampling theorem [15] to reconstruct \( G(\alpha) \) [or equivalently \( F(\mu) \)] from the available evenly spaced data in the \( \alpha \)-domain; i.e.,

\[
F(\mu) = \sum_{n=-K}^{K} G(n\Delta) \cdot \frac{\sin\left[\pi(\alpha - n\Delta)/\Delta\right]}{\pi(\alpha - n\Delta)/\Delta}.
\]

(5)

In this case, (5) is an approximation due to truncation errors [5]. This approach, Shannon Reconstruction (SR), was suggested by Papoulis for one-dimensional signals [5]. Stark et al. [10] and Clark et al. [12] utilized this scheme for two-dimensional signals.

Let \( g(z) \) be the IFT of \( G(\alpha) \) (\( z \) does not have the units of space in general.) It can be shown that (see Appendix A)

\[
g(z) = \int_{-X_0}^{X_0} f(x) \cdot b_1(z, -x) \, dx
\]

(6a)

where \( b_1(z, x) \) is called the blurring function and its FT with respect to \( x \) is defined by the following:

\[
b_1(z, \mu) = \left| T'(\mu) \right| \cdot \exp\left[j2\pi zT(\mu)\right].
\]

(6b)

Generally, the support of \( b_1(z, x) \), the IFT of an AM-FM signal, in the \( x \)-domain is not finite. Due to this fact, although \( F(\mu) \) is a band-limited function, however, \( G(\alpha) \) may not be a band-limited function in general. Hence, (5) would not yield exact results even in absence of truncation errors (see also Papoulis [5]).

B. Modified Shannon Reconstruction

We define

\[
P(\alpha) = \left| S'(\alpha) \right| \cdot G(\alpha) = \frac{F(\mu)}{\left| T'(\mu) \right|}
\]

(7)

Let \( p(z) \) be the IFT of \( P(\alpha) \). It can be shown that (see Appendix A)

\[
p(z) = \int_{-X_0}^{X_0} f(x) \cdot b_2(z, -x) \, dx
\]

(8a)

where the FT of the blurring function with respect to \( x \) is now defined by

\[
b_2(z, \mu) = \exp\left[j2\pi zT(\mu)\right].
\]

(8b)

It can be seen from (6b) and (8b) that (for the monotone \( T \) transformation, without loss of generality, we can assume that \( T'(\mu) = \left| T'(\mu) \right| \))

\[
b_1(z, \mu) = \frac{1}{j2\pi z} \cdot \frac{\partial B_1(z, \mu)}{\partial \mu},
\]

or equivalently,

\[
b_1(z, x) = \frac{x}{z} b_2(z, x).
\]

One might also use Shannon's sampling theorem to interpolate \( P(\alpha) \) from the available \( P(n\Delta) \) values. Then, one can recover \( G(\alpha) \) or \( F(\mu) \) from the estimated value of \( P(\alpha) \). This approach can be described by the following:

\[
F(\mu) = \left| T'(\mu) \right| \cdot \sum_{n=-K}^{K} \left| S'(n\Delta) \right| \cdot G(n\Delta) \cdot \frac{\sin\left[\pi(\alpha - n\Delta)/\Delta\right]}{\pi(\alpha - n\Delta)/\Delta}.
\]

(9)

We call this scheme the Modified Shannon Reconstruction (MSR). Note that both \( g(z) \) and \( p(z) \) have infinite supports. The accuracy of (5) or (9) depends upon the amount of smearing of \( f(x) \) by the blurring functions. Moreover, one cannot easily determine which one of the blurring functions causes more smearing and support expansion: \( b_1(z, x) \) is a high-pass version of \( b_2(z, x) \) in the \( x \)-domain, and \( b_2(z, x) \) is a high-pass version of \( b_1(z, x) \) in the \( z \)-domain.

We do not intend to study the relative merits of the SR and MSR methods in this paper. We brought out the MSR method due to the fact that it possesses certain characteristics of both the SR and UFR methods. Thus, we will have another closely related method to examine in our numerical study of the reconstruction methods.

C. Unified Fourier Reconstruction

Let \( i(x) \) be the indicator function of \( f(x) \); i.e.,

\[
i(x) = \begin{cases} 1 & \text{for } |x| \leq X_0 \\ 0 & \text{otherwise} \end{cases}
\]

and \( I(\mu) \) be the FT of \( i(x) \). The UFR scheme (see Appendix A) utilizes the following approximation to reconstruct the values of \( F(\mu) \):

\[
F(\mu) = \Delta \sum_{n=-K}^{K} G(\alpha_n) \cdot I[\mu - S(\alpha_n)] \cdot \left| S'(\alpha_n) \right|.
\]

(10)

Note that the interpolating kernel in (10) has two components. \( |S'(\alpha_n)| \) carries information regarding the measure of change in \( \mu \) for an incremental change of \( d\alpha \) in \( \alpha \). Thus, the task of \( |S'(\alpha_n)| \) in (10) is to normalize the density of the unevenly spaced available data. \( I(\cdot) \) is the Nyquist interpolating kernel that contains information about the support of \( f(x) \).

Consider a one-dimensional inverse system where \( \mu = \)
Suppose there are $2K + 1$ evenly spaced sample points in the $\alpha$-domain in the interval $[-1, 1]$. This set of data does not correspond to any specific inverse system. However, the model exemplifies a case of finer sampling at lower values of $|\mu|$, and coarser sampling at larger values of $|\mu|$ in the $\mu$-domain. We choose $f(x) = 1$ for $|x| \leq 10$, and zero otherwise (a rectangular pulse).

Fig. 1 shows the squared errors of the reconstructions obtained by the above-mentioned methods as a function of the number of available samples ($2K + 1$). Fig. 1(a) shows the sum of the squared error for the estimates obtained at 33 evenly spaced points in the interval $0 \leq \mu < 0.5$. Fig. 1(b) shows the same sum in the interval $0.5 \leq \mu < 1$. Fig. 1(c) depicts the sum of the errors in Fig. 1(a) and (b). Note that the SR and MSR methods show similar results. All three methods showed good reconstruction results for $2K + 1 > 100$. The UFR method showed lower errors for the high-frequency data when $60 < 2K + 1 < 100$. This type of behavior has also been observed in the two-dimensional inverse systems where the density of the available data was low at higher frequency regions. Consequently, the quality of the UFR's reconstruction at the object's edges was superior to those obtained by the filtered backprojection, bilinear interpolation, and two-dimensional SR methods [13], [14], [16]. Numerical comparisons of these reconstruction methods in certain two-dimensional inverse systems can be found in [9], [10], and [16].

The three reconstruction techniques described above, i.e., the SR, MSR, and UFR methods, are all ad hoc schemes. We indicated that the SR and MSR methods do not yield exact results. We will show that the UFR method also suffers from this problem. Moreover, the above methods do not necessarily yield the least-squares solution for the desired function. In fact, Chen and Allebach [11] showed that the least-squares solution has the following form:

$$F(\mu) = \sum_{\kappa} \alpha_\kappa \cdot I[\mu - S(\alpha_\kappa)]$$

where the coefficients $\alpha_\kappa$ should be obtained from the normal equations (see also Levi [4]). Nevertheless, the SR and UFR methods are generally used in multidimensional inverse problems due to their ease of implementation.

In the next section we present a study on the spectral (Fourier) characteristics of the available data. The study is similar to the spectral analysis of Pulse Position Modulation (PPM) given in [17]. This will help us to evaluate the UFR method and examine its properties with respect to the transform function and parameter variations of the imaging system.

III. FOURIER ANALYSIS OF THE AVAILABLE DATA

The IFT of an ideally (delta) and evenly spaced sampled data of a frequency domain signal consists of the IFT of the original signal repeated at the harmonics of the sampling rate. In this case, the IFT of the delta-sampled signal can be viewed as the output of a linear shift-invariant system; the impulse response of this system is a periodic delta function; the IFT of the original signal is the input to the system. Shannon's sampling theorem states that the original signal can be recovered by low-pass filtering its delta-sampled version provided that the sampling rate is greater than twice the highest frequency of the original signal.

In this section we formulate the problem of reconstruction from unevenly spaced sampled data in a framework similar to Shannon's sampling theorem for evenly spaced data. We show that the IFT of the unevenly spaced frequency domain data is the output of a linear shift-invari-
ant system when the input is the IFT of the original signal. The transfer function of the system is then identified using the parameters of the imaging system. This will help us to study the IFT of the available data and search for a scheme to recover the original signal.

A. Sampling System Identification

We denote the mapping of \( \alpha_n \) into the \( \mu \)-domain by

\[
\mu_n = S(\alpha_n) = S(n\Delta).
\]  

(11)

Thus, the set of available data may also be identified by \( F(\mu_n) = G(\alpha_n) \). We mentioned earlier that the transformation \( T(\cdot) \) is differentiable and one-to-one. Hence, the range of the available data in the \( \alpha \)-domain, i.e., \( [\alpha_{-K}, \alpha_K] \), maps into a contiguous interval in the \( \mu \)-domain, i.e., \( [\mu_{-K}, \mu_K] \). We denote the indicator (rectangular) function corresponding to the \( [\mu_{-K}, \mu_K] \) interval by \( W(\mu) \); i.e., \( W(\mu) = 1 \) for \( \mu \in [\mu_{-K}, \mu_K] \), and zero otherwise.

We define the following delta-sampled version of the available data in the \( \mu \)-domain:

\[
F_{\delta}(\mu) = \sum_{n=-K}^{K} F(\mu_n) \cdot \delta(\mu - \mu_n).
\]  

(12)

Using the window function \( W(\mu) \), the finite sum in (12) can be converted into an infinite sum

\[
F_{\delta}(\mu) = F(\mu) \cdot W(\mu) \cdot \sum_{n=-\infty}^{\infty} \delta(\mu - \mu_n).
\]  

(13)

It can be shown from (12) that

\[
f_{\delta}(x) = \sum_{n=-K}^{K} F(\mu_n) \cdot \exp(j2\pi\mu_n x).
\]  

(13a)

Our purpose is to reconstruct \( F(\mu) \) from the delta-sampled available data, i.e., \( F_{\delta}(\mu) \). Using (1)-(3), it can be shown that (see Theorem 5.4 in [18])

\[
\delta(\mu - \mu_n) = \frac{1}{S'(\alpha)} \cdot \delta(\alpha - \alpha_n) = \left| T'(\mu) \right| \cdot \delta\left| T(\mu) - n\Delta \right|.
\]  

(14)

Using (14) in (13) yields

\[
F_{\delta}(\mu) = F(\mu) \cdot W(\mu) \cdot \left| T'(\mu) \right| \sum_{n=-\infty}^{\infty} \delta\left| T(\mu) - n\Delta \right|.
\]  

(15)

We define the Jacobian Modified Transfer Function of the imaging system by

\[
H(\mu) = W(\mu) \cdot \sum_{n=-\infty}^{\infty} \delta\left( T(\mu) - n\Delta \right).
\]  

(16)

It can be shown from (16) that

\[
h(x) = \sum_{n=-K}^{K} \exp\left( j2\pi\mu_n x \right) \left| T'(\mu_n) \right|.
\]  

(16a)

With the help of Poisson's sum formula, the delta sum on the right side of (16) can be converted into a sum over exponentials that yields

\[
H(\mu) = \frac{1}{\Delta} W(\mu) \cdot \sum_{n=-\infty}^{\infty} \exp\left( j \frac{2\pi n}{\Delta} T(\mu) \right)
\]

\[
= \frac{1}{\Delta} W(\mu) + \frac{2}{\Delta} W(\mu) \sum_{n=1}^{\infty} \cos\left( \frac{2\pi n}{\Delta} T(\mu) \right).
\]  

(17)

Using (16), (15) may be rewritten as follows:

\[
F_{\delta}(\mu) = F(\mu) \cdot \left| T'(\mu) \right| \cdot H(\mu).
\]  

(18)

By inspecting (17) and (18), one can observe that \( F_{\delta}(\mu) \) is an amplitude modulated function of \( F(\mu) \cdot \left| T'(\mu) \right| \) and an exponential modulated function of \( T(\mu) \). Suppose that the cosine terms on the right side of (17) are highly fluctuating as compared to \( F(\mu) \cdot \left| T'(\mu) \right| \). In this case, an estimate of \( F(\mu) \cdot \left| T'(\mu) \right| \) can be obtained by low-pass filtering \( F_{\delta}(\mu) \). This scheme is similar to one of the approaches used in PPM to recover the transmitted message [17].

\[
\left| T'(\mu) \right| \text{ is not a band-limited function in general. Therefore, } F(\mu) \cdot \left| T'(\mu) \right| \text{ is not band-limited. Hence, low-pass filtering } F_{\delta}(\mu) \text{ might not yield a good estimate of } F(\mu) \cdot \left| T'(\mu) \right|.
\]

Fortunately, our problem is to recover \( F(\mu) \) which is band-limited. Moreover, \( \left| T'(\mu) \right| \) is a known and well-defined function (not a delta sequence). Hence, the effects of the Jacobian function in the delta-sampled available data can be accounted for through the normalization of \( F_{\delta}(\mu) \) by \( \left| T'(\mu) \right| \).

We define the Jacobian Modified Response of the imaging system in the \( \mu \)-domain by dividing both sides of (18) with \( \left| T'(\mu) \right| \)

\[
R(\mu) = \frac{F_{\delta}(\mu)}{\left| T'(\mu) \right|} = F(\mu) \cdot H(\mu).
\]  

(19)

Using (12) and (19), one can show that

\[
r(x) = \sum_{n=-K}^{K} \frac{F(\mu_n)}{\left| T'(\mu_n) \right|} \cdot \exp\left( j2\pi\mu_n x \right).
\]  

(19a)

It is clear from (19) that \( r(x) \), the IFT of \( R(\mu) \), is the response of a linear shift-invariant system with transfer function \( H(\mu) \) when the input is \( f(x) \) (see Fig. 2). In this case, \( R(\mu) \) is an amplitude modulated function of \( F(\mu) \) and an exponential modulated function of \( T(\mu) \). Thus, if the rates of fluctuations of the cosine terms in (17) were sufficiently larger than \( X_0 \) [the bandwidth of \( F(\mu) \)], then \( F(\mu) \) could be recovered by low-pass filtering \( R(\mu) \).

Note that the presence of \( \left| T'(\mu) \right| \) on the right side of (18) is due to the decomposition shown in (14). Obviously, this decomposition is not unique. In this case,

\[\text{We will examine the case when } T'(\mu) \text{ is unknown in Section V.}\]
other types of decompositions for \( \delta(\mu - \mu_n) \) might reveal other solutions for our problem. However, as we show in the next section, the Jacobian modified response has certain characteristics that make it a suitable choice in this reconstruction problem.

**B. Spectrum of the Jacobian Modified Response**

Using the expression given for the Jacobian modified transfer function in (17), the Jacobian modified response may be written as follows:

\[
R(\mu) = \frac{1}{\Delta} F(\mu) + \frac{1}{\Delta} E(\mu)
\]

(20)

where \( E(\mu) \) is defined to be

\[
E(\mu) = F(\mu) \cdot [\Delta H(\mu) - 1]
\]

(21a)

with

\[
E_1(\mu) = F(\mu) \cdot [\Delta H(\mu) - 1]
\]

(21b)

and

\[
E_2(\mu) = 2F(\mu) \cdot W(\mu) \cdot \sum_{n=1}^{\infty} \cos \left(\frac{2\pi n\Delta}{\Delta} \cdot \mu \right)
\]

(21c)

The IFT of the Jacobian modified response of the imaging system, i.e., \( \mathcal{F}(\mu) \), is a linear combination of \( f(x) \) and \( e(x) \) [see (20)]. Thus, one possible approach to recover \( F(\mu) \) or \( f(x) \) is a linear processing of \( R(\mu) \) that results in the suppression of the effects of \( e(x) \) in \( r(x) \) with minimum distortion in the structure of \( f(x) \). This could be achieved if the support of \( e(x) \) resided outside the \([X_0, X_0]\) interval. We now examine \( e_1(x) \) and \( e_2(x) \) to find out if this condition could be satisfied.

1) **Analysis of \( E_1(\mu) \):** Consider the window \([ W(\mu) - 1 \] on the right side of (21b). The presence of this term signifies the finite bandwidth of the imaging system. In this case, \( E_1(\mu) = -F(\mu) \) [see (21b)] in the region of the \( \mu \)-domain where no data point is available. For most imaging systems, this corresponds to the high spatial frequency components of \( F(\mu) \). This is a determining factor as far as the resolution of the imaging system is concerned. It is often difficult to extrapolate the high spatial frequency contents. Thus, one should expect to see the effects of \( W(\mu) \) rectangular windowing (truncation errors, Gibbs phenomenon [5]) in the final reconstructed function. This implies that the contribution of \( e_1(x) \) in \( r(x) \) cannot be separated from \( f(x) \) (however, one may reduce these effects by windowing techniques). Consequently, a distortionless reconstruction cannot be obtained.

2) **Analysis of \( E_2(\mu) \):** The presence of \( \cos \left(\frac{2\pi n\Delta}{\Delta} \cdot \mu \right) \) \( T(\mu) \), for \( n \geq 1 \), in the expression for \( E_2(\mu) \) [see (21c)], suggests that \( e_2(x) \) can be a high-spatial function. It is clear that an almost distortionless reconstruction (truncation errors are still present) is possible provided that the support of \( e_1(x) \) does not overlap with the support of \( f(x) \). This requirement can be viewed as the condition to avoid aliasing in the reconstructed data.

When \( T(\mu) \) is a linear transformation of the form \( T(\mu) = a\mu \), the available data points in the \( \mu \)-domain, i.e., \( \mu_n \)'s, are evenly spaced \((\mu_{n+1} - \mu_n = \Delta/a, \text{for all } n)\). In this case, \( E_2(\mu) \) is simply a linearly modulated form of \( F(\mu) \); i.e.,

\[
E_2(\mu) = 2F(\mu) \cdot W(\mu) \cdot \sum_{n=1}^{\infty} \cos \left(\frac{2\pi n\Delta}{\Delta} \cdot \mu \right)
\]

and \( e_2(x) \) is composed of shifted versions (linear harmonics) of \( f(x) \) smeared (convolved) by \( w(x) \), i.e.,

\[
e_2(x) = w(x) * \sum_{n=0}^{\infty} f(x - na) \]

This is Shannon’s sampling theorem where (by neglecting the effects of \( w(x) \), the truncation errors) the support of \( e_2(x) \) does not coincide with the support of \( f(x) \) if \(|a|/\Delta > 2X_0\).

The Fourier analysis of \( E_2(\mu) \) becomes a formidable task when \( T(\mu) \) is a nonlinear transformation. In this case, the support of \( e_2(x) \) (even in absence of \( w(x) \) effects) is not finite in general. This is due to the nonlinearly (phase) modulated terms in (21c). However, one may define a finite effective bandwidth for the cosine terms in (21c). For instance, this finite effective bandwidth can be the 98 percent energy bandwidth or the bandwidth defined by the Carson’s rule [17]. Then, one can examine the conditions under which the finite effective bandwidth of \( E_2(\mu) \), the nonlinear harmonics, would reside outside the support of \( f(x) \).

Consider the rectangular pulse/cubic transformation example that we cited earlier. Fig. 3(a) and (b) shows the distributions of \( f(x) \) [see (13a)] and \( r(x) \) [see (19a)], respectively, for \( 2K + 1 = 121 \). In this case, one can identify the object function and the two error terms in
\( r(x) \), \( f_{k}(x) \) contains a smeared version of \( f(x) \); thus, one cannot recover the desired signal by simply low-pass filtering the delta sampled signal. As we mentioned earlier, Horiuchi [6] showed a sampling scheme, based on the exact knowledge of the instantaneous bandwidth of \( F(\mu) \), where a shift-varying filter was used to recover \( F(\mu) \) from \( F_{k}(\mu) \).

C. Minimum Sampling Rate

In this section, we relate the spatial dispersion of the nonlinear harmonics, i.e., \( e_{2}(x) \), to the transform function and the parameters of the inverse system. This will help us to determine the minimum sampling rate that reduces the nonlinear aliasing and, consequently, yields accurate reconstructions. This topic requires a separate analysis for the multidimensional inverse problems.

1) One-Dimensional Systems: We denote the 98 percent energy bandwidths of \( T(\mu) \) and \( W(\mu) \) by \( X_{T} \) and \( X_{W} \), respectively. Using the Carson’s rule, the minimum spatial dispersion of the cosine functions in (21c) can be found to be

\[
\text{minimum}_{\mu} \left[ \frac{n}{\Delta} T'(\mu) \right] = X_{T}, \quad \text{for} \quad \mu \in [\mu_{-K}, \mu_{K}].
\]

We denote the minimum value of \( T'(\mu) \) in the region of the available data by \( \sigma \). The minimum spatial dispersion for the sum of the cosine functions in (21c) corresponds to the case of \( n = 1 \); i.e.,

\[
\frac{\sigma}{\Delta} = X_{T}.
\]

In the expression for \( E_{2}(\mu) \) [see (21c)], the cosine functions are amplitude modulated by \( F(\mu) \cdot W(\mu) \). In this case, the minimum spatial dispersion of \( e_{2}(x) \) is approximately equal to

\[
\frac{\sigma}{\Delta} = X_{0} - X_{T} - X_{W}.
\]

Hence, the support of \( e_{2}(x) \) resides outside the support of \( f(x) \) if

\[
\frac{\sigma}{\Delta} = X_{0} - X_{T} - X_{W} > X_{0}
\]

or

\[
\frac{\sigma}{\Delta} > 2X_{0} + X_{T} + X_{W} = 2R_{0}. \quad (22a)
\]

The maximum value of \( \Delta \) (the minimum sampling rate) that yields accurate reconstructions can be obtained from (22a).

The effective bandwidth of \( W(\mu) \) is commonly assumed to be equal to the first two sidelobes of \( w(x) \). This bandwidth is much smaller than the bandwidth of \( F(\mu) \), i.e., \( X_{0} \), when most of the energy of \( F(\mu) \) resides in \([\mu_{-K}, \mu_{K}]\) [5]. In many practical inverse systems, this condition can be satisfied. Moreover, when the phase modulated cosine functions in (21c) correspond to wide-band an-
ingle modulation (i.e., their instantaneous frequency, $nT' (\mu)/\Delta$, is much greater than the bandwidth of $T(\mu)$, $X_\mu$), one may neglect the effects of $X_\mu$ in (22a). Thus, (22a) can be approximated by the following:

$$\sigma / \Delta > 2X_\mu.$$  \hspace{1cm} (22b)

Consider the cubic transformation example. In this case, $\sigma = \frac{3}{2}$ and $\Delta = 4/2K$. Using (22b), one can show that $2K > 12X_\mu$. Thus, for the rectangular pulse where $X_\mu = 10$, we should have $2K > 120$. In this case, Fig. 3(b) ($2K + 1 = 121$) corresponds to the case when the minimum allowable number of samples is used. Fig. 3(c) and (d) depicts the distributions of $r(x)$ for slightly undersampled ($2K + 1 = 91$) and oversampled ($2K + 1 = 151$) cases, respectively.

It can be shown from (22) that, for almost distortionless reconstruction, the Jacobian modified impulse response, i.e., $h(x)$ [see (16a)], should approximate the sampling function $w(x)/\Delta$ for $|x| \leq 2R_0$. However, $h(x)$ may have substantial energy beyond this region. Fig. 4 shows the distributions of $h(x)$ for three different values of the number of samples. The maximum allowable values of $X_\mu$, using (22b), are also indicated in the figures. The 98 percent energy bandwidth of $T(\mu)$ is approximately equal to two, and the first two sidelobes of $w(x)$ are equal to one. In this case, (22a) yields $2K > 12X_\mu + 18$. Using this constraint, we have for (a) $2K = 60$, $X_\mu < 3.5$; (b) $2K = 120$, $X_\mu < 8.5$; and (c) $2K = 240$, $X_\mu < 18.5$.

When $T(\mu) = \alpha\mu$, then $\sigma = \alpha$, and the result of (22b) is identical to the Nyquist rate. Suppose we can write the following approximation:

$$\Delta = \alpha_{n+1} - \alpha_n = T(\mu_{n+1}) - T(\mu_n) \approx (\mu_{n+1} - \mu_n) \cdot T'(\mu_n).$$

Then, due to the fact that $\sigma \leq T' (\mu)$, we can conclude from the above that

$$\Delta > (\mu_{n+1} - \mu_n) \cdot \sigma.$$  \hspace{1cm} (22c)

Comparing this result to inequality (22b), one can show that

$$\mu_{n+1} - \mu_n < \frac{1}{2X_\mu}$$

for all values of $n$.

Thus, a possible interpretation of (22b) can be that the Nyquist rate should be satisfied in the region of the available data for accurate reconstruction (although $\mu_{n+1} - \mu_n$ may vary with $n$). Thus, within the approximations that we made regarding the support of the error terms, one may state that a band-limited signal can be approximately recovered from its unevenly spaced sampled data provided that the variable sampling rate satisfies the Nyquist criterion. This conclusion can also be found in the work of Landau [7].

\footnote{Our results have indicated that this is a valid assumption for most differentiable $T(\cdot)$ transformations when $\Delta \ll |\alpha_e - \alpha_s|$.}

2) Two-Dimensional Systems: Consider a two-dimensional function $f(x, y)$ that is zero outside the disk of radius $R_0$ centered at $x = 0$. Let $F(\mu, \lambda) = \int f(x, y) \cdot e^{-j(\mu x + \lambda y)} dx dy$, $\alpha = T_1(\mu, \lambda)$, and $\beta = T_2(\mu, \lambda)$, where $T_1$ and $T_2$ are differentiable transformations. Suppose $G(\alpha, \beta) = F(\mu, \lambda)$. The available data comprise a finite set of equally spaced values of $G$ in the $(\alpha, \beta)$ domain, e.g., $\alpha_0 = m\Delta_1$ and $\beta_0 = n\Delta_2$. We are interested in determining the maximum values of $\Delta_1$ and $\Delta_2$ for accurate reconstruction of $f(x, y)$.

In this case, the nonlinear harmonics (angle modulated error term) can be shown to have the following form:
\[ E_2(\mu, \lambda) = F(\mu, \lambda) \cdot W(\mu, \lambda) \cdot \sum_{(m, n) \neq (0, 0)} \exp \left[ j2\pi \frac{mT_1(\mu, \lambda)}{\Delta_1} + j2\pi \frac{nT_2(\mu, \lambda)}{\Delta_2} \right] \]

where \(W\) is the indicator function for the region of the available data in the \((\mu, \lambda)\) domain. We denote the maximum 98 percent energy bandwidth of the linear combinations of \(T_1(\mu, \lambda)\) and \(T_2(\mu, \lambda)\) by \(D : X_T\), and the 98 percent energy bandwidth of \(W(\mu, \lambda)\) by \(D : X_W\). The support of \(e_3(x, y)\) would reside outside \(D : X_0\) if the center (instantaneous) frequencies of the PM waves\(^3\) on the right side of the above equation satisfy

\[
\begin{align*}
\frac{m\nabla T_1(\mu, \lambda)}{\Delta_1} + \frac{n\nabla T_2(\mu, \lambda)}{\Delta_2} > 2X_0 + X_T + X_W = 2R_0,
\end{align*}
\]

for all integer values of \((m, n) \neq (0, 0)\), where \(\nabla\) is the gradient operator in the \((\mu, \lambda)\) domain and \(|A|\) corresponds to the magnitude of the vector \(A\). After some rearrangements, (23) can be rewritten as follows:

\[
m^2Z_1^2 + n^2Z_2^2 - 2mnZ_1Z_2 \cos \phi > 1 \tag{24a}
\]

where

\[
Z_1(\mu, \lambda) = \frac{\nabla T_1(\mu, \lambda)}{2\Delta_1R_0} \tag{24b}
\]

\[
Z_2(\mu, \lambda) = \frac{\nabla T_2(\mu, \lambda)}{2\Delta_2R_0} \tag{24c}
\]

and \(\phi(\mu, \lambda)\) is the angle between the vectors \(\nabla T_1(\mu, \lambda)\) and \(\nabla T_2(\mu, \lambda)\).

Consider the following two cases.

i) \((m, n) = (1, 0)\): inequality (24a) becomes

\[
Z_1 > 1 \tag{25a}
\]

and

ii) \((m, n) = (0, 1)\): inequality (24a) becomes

\[
Z_2 > 1. \tag{25b}
\]

Inequalities (25a)–(25b) [with (24b)–(24c)] provide (maximum value) solutions for \(\Delta_1\) and \(\Delta_2\). Unfortunately, one cannot conclude that with these values of \(\Delta_1\) and \(\Delta_2\), (24a) can be satisfied for all \((m, n)\). The study of (24a) for all values of \((m, n)\) to determine \(\Delta_1\) and \(\Delta_2\) is a difficult task. The problem becomes more complicated for three and higher dimensional inverse systems. A practical approach to this problem might be to search for functional characteristics in the transform function that could simplify our analysis.

For instance, suppose \(\left| \nabla T_1 \right|\) and \(\left| \nabla T_2 \right|\) assume their minimum values at \((\mu_0, \lambda_0)\), and \(\phi(\mu_0, \lambda_0)\) \(= \phi_0\). Moreover, we know that \(\phi(\mu, \lambda) \in [\phi_0, \pi - \phi_0]\) in the region of the available data.\(^4\) Using analytic geometric

\(^3\)The center frequency of a two-dimensional PM wave, e.g., \(\sqrt{2\pi A(\mu, \lambda)}\), is defined by \(\sqrt{|A(\mu, \lambda)|}\); this is a radial distance measure from the origin in the spatial \((r, y)\) domain.

\(^4\)This is the case in diffraction tomography [14, 16].

Fig. 5. A collection of elliptical contours defined by (24a) for \(\phi_0 = \pi/6\), \(m = 1, \ldots, 10\), and \(n = 1, \ldots, 10\).

\[
\frac{2\pi}{3} \geq |\phi| \geq \frac{\pi}{3}, \tag{26}
\]

hence, \(|\cos \phi| \leq 0.5\). It can be shown that for the class of two-dimensional transformations that (26) is satisfied, one can use (25) to determine the maximum values of \(\Delta_1\) and \(\Delta_2\).

Consider the following polar transformation that appears in parallel beam straight-path tomography:

\[
\mu = \alpha \cos \beta \quad \text{and} \quad \lambda = \alpha \sin \beta,
\]

\(\alpha\) is the spatial frequency pair for the receiver (sensor) position and \(\beta\) is the propagation angle. Suppose there are \(M\) values of \(\alpha_n\) evenly spaced in \([-\alpha_0, \alpha_0]\) and \(N\) values of \(\beta_n\) evenly spaced in \([-\pi/2, \pi/2]\). For this example, one can show that \(\phi = \pi/2\) (i.e., (26) is satisfied). Hence, we can use (24) to determine \(\Delta_1\) and \(\Delta_2\). The final results are as follows:

\[
\Delta_1 < \frac{1}{2R_0} \tag{26a}
\]

and

\[
\Delta_2 < \frac{1}{\sqrt{\mu^2 + \lambda^2} 2R_0}. \tag{26b}
\]

Suppose we have \(R_0 = X_0\). Moreover, we know that \(\Delta_2 = \pi/N\) and \(\sqrt{\mu^2 + \lambda^2} = |\alpha|\). Using these in (26b) yields

\[
N > 2\pi |\alpha| X_0, \tag{26c}
\]
the worst case corresponds to $|\alpha| = \alpha_0$. The minimum number of projections in computerized tomography has also been estimated through other techniques (e.g., see [2] and [14]). The result is identical to (26c).

IV. Reconstruction

In the previous sections we studied the Jacobian modified response of the imaging system and the conditions for the recovery of the desired function. We now present schemes to recover $f(x)$ through low-pass filtering the Jacobian modified response of the system.

A. Spatial Domain Reconstruction

We mentioned that the distribution of $r(x)$ in the interval $[-X_0, X_0]$ is a good estimate of $f(x)$ if the inequality (22) is satisfied. Thus, it is sufficient to evaluate the inverse spatial Fourier transform of $R(\mu)$ within the support of the desired spatial domain function to recover $f(x)$. This corresponds to the following operation:

$$f(x) \approx \Delta \int_{-\infty}^{\infty} R(\mu) \cdot \exp(j2\pi \mu x) \, d\mu$$

for $|x| \leq X_0$. \hfill (27)

Using (12) and (19) in (27) yields

$$f(x) \approx \Delta \int_{-\infty}^{\infty} \frac{F(\mu)}{|T'(\mu)|} \cdot \sum_{n=-K}^{K} \delta(\mu - \mu_n) \cdot \exp(j2\pi \mu_n x) \, d\mu.$$ \hfill (28)

After reversing the order of integration and summation in (28), one obtains

$$f(x) \approx \Delta \sum_{n=-K}^{K} \int_{-\infty}^{\infty} \frac{F(\mu)}{|T'(\mu)|} \cdot \delta(\mu - \mu_n) \cdot \exp(j2\pi \mu_n x) \, d\mu$$

$$= \Delta \sum_{n=-K}^{K} \frac{F(\mu_n)}{|T'(\mu_n)|} \cdot \exp(j2\pi \mu_n x). \hfill (29)$$

Equation (29) can be used to reconstruct $f(x)$ at some desired values of $x$. However, this scheme is computationally intensive, especially in the case of multidimensional reconstruction problems, due to the presence of the complex exponential term on the right side of (29). Note that this exponential term represents the transformation of the spatial frequency domain data to the spatial domain. Thus, it might be more convenient to reconstruct evenly spaced values of $F(\mu)$ from the available data. Then one can use fast discrete Fourier transform routines to recover the desired spatial domain function. This is discussed in the next section.

B. Frequency Domain Reconstruction

Based on our previous discussions, we can write

$$f(x) = \Delta r(x) \cdot i(x). \hfill (30)$$

Taking the spatial FT of both sides of (30) yields

$$F(\mu) = \Delta \int_{-\infty}^{\infty} R(\eta) \cdot I(\mu - \eta) \, d\eta. \hfill (31)$$

Using the expression for the Jacobian modified response term from (19) and (12) in (31) yields

$$F(\mu) = \Delta \left[ \int_{-\infty}^{\infty} \frac{F(\eta)}{|T'(\eta)|} \right] \cdot \sum_{n=-K}^{K} \delta(\eta - \mu_n) \cdot I(\mu - \eta) \, d\eta. \hfill (32)$$

Reversing the order of integration and summation in (32) and after some rearrangements, one obtains

$$F(\mu) \approx \Delta \sum_{n=-K}^{K} \int_{-\infty}^{\infty} \frac{F(\eta)}{|T'(\eta)|} \cdot I(\mu - \eta) \cdot \delta(\eta - \mu_n) \, d\eta$$

$$= \Delta \sum_{n=-K}^{K} \frac{F(\mu_n)}{|T'(\mu_n)|} \cdot I(\mu - \mu_n). \hfill (33)$$

With the help of (2)–(4) and a change of $\mu_n$ to $\alpha_n$, one can rewrite (33) as follows:

$$F(\mu) \approx \Delta \sum_{n=-K}^{K} G(\alpha_n) \cdot \left[ I[\mu - S(\alpha_n)] \cdot \left| S'(\alpha_n) \right| \right] \hfill (34)$$

where (34) is identical to (10) that is used in the UFR method to reconstruct $F(\mu)$.

The frequency domain reconstruction requires less computational time than the spatial domain one (see [13] and [16]). However, the reconstruction based on (34) is identical to the reconstruction using (29). This is due to the fact that both (29) and (34) yield the distribution of $r(x)$ in the interval $|x| \leq X_0$. Note that the spatial domain reconstruction outlined here is different from the spatial domain schemes that were cited in [13], such as filtered backprojection in straight-path tomography. Filtered backprojection can be viewed as a computationally efficient way of implementing the spatial domain technique, i.e., (29) (see Appendix B). However, this is achieved through further approximations and interpolations. This obviously makes the filtered backprojection’s reconstruction less accurate than the reconstruction obtained using (29) or (34). As we mentioned earlier, [16] provides a numerical study of the filtered backprojection, bilinear interpolation, and UFR methods.

Fig. 6(a) shows the reconstructed object function for the rectangular pulse/cubic transformation example using the UFR method ($2K + 1 = 121$). Fig. 7(a) is the UFR reconstruction of a two-dimensional object function (head phantom, see [16]). The reconstructed image is $128 \times 128$ with 94 gray levels. The available data were obtained from the polynomial transformation (parallel beam straight-path tomography) with $X_0 = 1$, $\alpha_0 = 32$, $M = 128$, and $N =$
64. Using (26c), it can be shown that the spatial frequency region specified by $10 < \sqrt{\mu^2 + \lambda^2} \leq 32$ suffers from nonlinear aliasing; in fact, based on (26c), the minimum number of projections should be $N = 201$. It turns out that most of the energy of the head phantom lies in $\sqrt{\mu^2 + \lambda^2} < 10$; thus, the effect of nonlinear aliasing errors (the streaks seen outside the reconstructed phantom) is not significant. Some other applications of UFR and comparisons to some other known methods can be found in [13], [14], and [16].

In both of the above examples, the Jacobian function, $|T'|$, is infinity at the dc point on the frequency plane. This corresponds to a high density of available data at low spatial frequencies. In the filtered backprojection algorithm, this has been known to cause *dishing* effects (see [16]) and large errors in the dc value of the reconstructed image. In the examples that we studied, the UFR method did not show any dishing effect. However, it yielded erroneous results for the dc value when the FT of the object function was highly fluctuating around the dc point. This was the case for the head phantom. On the other hand, the estimate of the rectangular pulse's dc value was accurate. We also examined transformations with $T'(\mu) = 0$, for certain values of $\mu$; this results in severe undersam-
V. RECONSTRUCTION FOR UNKNOWN TRANSFORMATIONS

Consider (10) that is used in the UFR method to reconstruct $F(\mu)$. Suppose the imaging system produces the values of the sampled points, i.e., $\mu_n = S(\alpha_n)$, and the functional values at the sampled points, i.e., $F(\mu_n) = G(\alpha_n)$. However, the exact knowledge of the transformation $T(\cdot)$ may not be available. By inspecting (10), one can observe that the only information required to construct the UFR sum in (10), in addition to $\mu_n$ and $F(\mu_n)$, is the Jacobian function, $|S'(\alpha_n)|$, or equivalently, $|T'(\mu_n)|$; this is not the case for the SR method where the knowledge of the $T(\cdot)$ transformation is required to construct (5). We now present a scheme to estimate the Jacobian function from the knowledge of $\mu_n$ when the $T(\cdot)$ transformation is unknown.

We choose $F(\mu) = 1$; i.e., the object function is the impulse function, $f(x) = \delta(x)$. In this case, (13) can be written as follows:

$$F_1(\mu) = \sum_{n=-K}^{K} \delta(\mu - \mu_n)$$  \hspace{2cm} (35)

and

$$f_1(x) = \sum_{n=-K}^{K} \exp(j2\pi \mu_n x).$$ \hspace{2cm} (35a)

$f_1(x)$ in (35a) is the delta-sampled response of the inverse system when the input is an impulse function. Note that the right side of (35a) can be computed without the knowledge of the transform function and its Jacobian. Fig. 8 shows the distribution of $f_1(x)$ for the cubic transformation example.

Substituting $F(\mu) = 1$ in (18) yields

$$F_1(\mu) = |T'(\mu)| \cdot H(\mu).$$ \hspace{2cm} (36)

Equation (36) has the same form as (19) where $r(x)$ was the response of the imaging system with input $f(x)$. Similarly, from (36) one can view $f_1(x) [f(x) = \delta(x)]$ to be the response of the imaging system when the input is the IFT of $|T'(\mu)|$. However, we showed in (35) that $F_1(\mu)$ or $f_1(x)$ can be computed when the transform function is unknown. Hence, $|T'(\mu)|$ could be recovered by low-pass filtering $F_1(\mu)$ if $|T'(\mu)|$ were band-limited in, e.g., $[-X_f, X_f]$, and “no aliasing” conditions could be satisfied (i.e., (22) with $X_f$ replaced by $X_f$).

Let the indicator function for $[-X_f, X_f]$ be $i_f(x)$. Then, an estimate of the Jacobian function can be obtained by the following:

$$|T'(\mu)| \approx \Delta \int_{-\infty}^{\infty} F_1(\eta) \cdot i_f(\mu - \eta) d\eta.$$ \hspace{2cm} (37)

This operation is identical to one of the schemes used in PPM to recover the transmitted message (see [17]). Using (35) in (37) yields

$$|T'(\mu)| \approx \Delta \sum_{n=-K}^{K} \delta(\eta - \mu_n) I_f(\mu - \eta) d\eta$$

$$= \Delta \sum_{n=-K}^{K} I_f(\mu - \mu_n).$$ \hspace{2cm} (38)

Generally, $|T'(\mu)|$ is not a band-limited function. Hence, one should choose a finite value for $X_f$, e.g., the 98 percent energy bandwidth of $|T'(\mu)|$. In this case, this bandwidth should be known a priori. The IFT of $|T'(\mu)|$ commonly has most of its energy at lower spatial values due to the fact that $T(\mu)$ is a differentiable function. Hence, one can obtain a good estimate of $X_f$ by inspecting the distribution of $f_1(x)$. Consider $f_1(x)$ for the cubic transformation in Fig. 8. Our experiment showed that $10 \leq X_f \leq 16$ would yield good estimates of the Jacobian function.

We now consider the original reconstruction problem, i.e., recovering the unknown object function from the knowledge of its unevenly spaced sampled points and the estimated Jacobian function. Note that the Jacobian estimate of (38) is inverted and then used in the UFR scheme [deconvolving $R(\mu)$ from $F_1(\mu)$; see (10) and (19)]. In this case, the errors in the estimate of (38) can have serious effects in the reconstructed function when the actual Jacobian function is very small or very large for certain values of $\mu$. This was the case in both the cubic and polar transformations; the transformations’ Jacobians are very large at lower spatial frequencies. Fig. 6(b) shows the reconstruction of the rectangular pulse from the cubic transformation data when the Jacobian function was estimated using (38). Fig. 7(b) shows the reconstructed head phantom from the polar transformation data when (38) was used to estimate the Jacobian. The reconstructed functions in Figs. 6(b) and 7(b) contained significant errors at lower spatial frequencies; these errors caused dishing effects in the reconstructions.

Fig. 6(c) is the reconstructed pulse function when $\mu_n$
values are chosen from a uniform \([-1, 1]\) distribution. The result shows a distorted pulse; however, this is misleading; the reconstructions of more complex object functions contained large errors. This can be attributed to the fact that there were large intervals in the \(\mu\) domain that contained no data points. Fig. 7(c) shows the reconstructed head phantom using randomly located polar data points: there were 64 radial values, \(\alpha_m\), chosen from the uniform \([0, 32]\) distribution; and the angular values, \(\beta_n\), were the same as in Fig. 7(a) and (b) (not random). This set of data also suffered from large regions of no data points.

The purpose of this section was to show our basic approach in the problem of reconstruction for unknown transformations; however, our work is not complete. We are currently studying other means of estimating the Jacobian function that yield reliable reconstructions. One approach is based on estimating \(F(\alpha) \cdot \{T'(\mu)\}\) by low-pass filtering \(F(\alpha)\) (see (18)); this estimate is then divided by the estimate of \(\{T'(\mu)\}\), i.e., (38), to obtain an estimate of \(F(\mu)\). There are various factors (such as the relationships between \(X_0\) and \(X_1\)) that are crucial in the success of these reconstruction schemes; some could be controlled to reduce nonlinear aliasing effects. These will be discussed in the future.

VI. CONCLUSIONS

A method for reconstructing band-limited functions from a finite number of their unevenly spaced data was introduced. It was shown that the scheme utilizes a filtering of the available data, which is similar to Shannon’s interpolation for evenly spaced data, to reconstruct the desired function. We presented algorithms to implement the reconstruction scheme. In the process, we showed that the UFR method is a computationally efficient way of implementing the scheme. We also presented an approach to estimate the Jacobian function when the transform function is unknown. The examples showed the merits of the schemes that were discussed in the paper.

Appendix A

1) Derivation of (6): From the inverse Fourier transform integral we have

\[
g(z) = \int_{-\infty}^{\infty} G(\alpha) \cdot \exp(j2\pi\alpha z) \, d\alpha. \tag{A1}
\]

Using (1)-(4), one can make a variable transformation from \(\alpha\) to \(\mu\) in (A1) that yields

\[
g(z) = \int_{-\infty}^{\infty} F(\mu) \cdot \exp\left\{j2\pi T'(\mu)\right\} \cdot \left|T'(\mu)\right| \, d\mu. \tag{A2}
\]

Substituting for \(F(\mu)\) in (A2) in terms of the Fourier integral of \(f(x)\) and after some rearrangements, one obtains

\[
g(z) = \int_{-\infty}^{X_0} f(x) \cdot \left[ \int_{-\infty}^{\infty} \left|T'(\mu)\right| \cdot \exp\left[j2\pi z T'(\mu) - j2\pi x \mu\right] \, d\mu \right] \, dx, \tag{A3}
\]

(A3) is equivalent to (6).

2) Derivation of (8): The procedure is similar to the one shown in (A1)-(A3) with \(g(z)\) replaced by \(p(z)\).

3) Derivation of (10): We can write the following identity equation:

\[
f(x) = i(x) \cdot f(x). \tag{A4}
\]

Taking the spatial Fourier transform of both sides of (A4) yields

\[
F(\mu) = \int_{-\infty}^{\infty} F(\eta) \cdot I(\mu - \eta) \, d\eta. \tag{A5}
\]

With the help of (1)-(4), we can make a variable transformation from \(\eta\) to \(\alpha\) in the integral of (A5) to obtain

\[
F(\mu) = \int_{-\infty}^{\infty} G(\alpha) \cdot I(\mu - S(\alpha)) \cdot \left|S'(\alpha)\right| \, d\alpha. \tag{A6}
\]

In practice, the integral in (A6) over \(\alpha\) values is approximated by a sum over the available \(\alpha_n\) values that yields (10).

APPENDIX B

Consider the polar transformation data in parallel beam straight-path tomography (see Section III-C-2). The spatial domain reconstruction, i.e., (29), can be shown to have the following form:

\[
f(x, y) = \Delta_x \Delta_x \sum_{n=1}^{N} \sum_{m=1}^{M} |a_m| G(\alpha_m, \beta_n) \cdot \exp\left[j2\pi \alpha_n (x \cos \beta_n + y \sin \beta_n)\right]. \tag{B1}
\]

The sum over \(m\) values in (B1) has the form of a Fourier sum. For a given \(\beta\), we define

\[
Q_{\beta}(\tau) = \text{IFT}_{\beta}\left[|a| \cdot G(\alpha, \beta)\right]
\]

where \(\tau\) is the inverse Fourier domain for \(\alpha\). Thus, an efficient way to compute (B1) is to obtain \(Q_{\beta}(\tau_n)\) from \(|a_m| \cdot G(\alpha_m, \beta_n)\) using FFT’s (\(\tau_n\)’s are equally spaced). Then, one can use interpolation techniques to estimate \(Q_{\beta}(\tau)\), at \(\tau = x \cos \beta_n + y \sin \beta_n\), from the available \(Q_{\beta}(\tau_n)\) values. This approach is identical to the filtered backprojection reconstruction method [2], [16].

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