Reconnaissance with Slant Plane Circular SAR Imaging

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Abstract—This paper presents a method for imaging from the slant plane data collected by a synthetic aperture radar (SAR) over the full rotation or a partial segment of a circular flight path. A Fourier analysis for the Green's function of the imaging system is provided. This analysis is the basis of an inversion for slant plane circular SAR data. The reconstruction algorithm and resolution for this SAR system are outlined. It is shown that the slant plane circular SAR, unlike the slant plane linear SAR, has the capability to extract three-dimensional imaging information of a target scene. The merits of the algorithm are demonstrated via a simulated target whose ultra wideband foliage penetrating (FOPEN) or ground penetrating (GPEN) ultrahigh frequency (UHF) radar signature varies with the radar's aspect angle.

I. INTRODUCTION

ULTRAHIGH frequency (UHF) and very high frequency (VHF) radars have the ability to penetrate foliage and soil. Due to this fact, these radars are currently being investigated for reconnaissance with synthetic aperture radar: foliage-penetrating (FOPEN) and ground-penetrating (GPEN) synthetic aperture radars (SAR's) [18]. The FOPEN/GPEN SAR systems are also referred to as ultra-wideband (UWB), since the bandwidth of the transmitted signal, which should be large enough to provide sufficient range resolution (e.g., 250 MHz), is comparable to a UHF/VHF carrier (e.g., 300 MHz).

FOPEN/GPEN SAR research has been mainly focused on imaging and target recognition algorithms when a stripmap data collection (side-looking radar) is used. In these systems, a radar-carrying aircraft moves along a line while it illuminates the target scene; we refer to these imaging systems as linear SAR. Accurate Fourier-based SAR imaging algorithms have been successfully used for these linear SAR systems [1]-[3], [7]-[9]. One of the problems with a linear SAR system in the reconnaissance problems of FOPEN/GPEN SAR is that a target SAR signature, with sufficiently high signal-to-noise power ratio, can be measured over a limited aspect angle interval, e.g., ±45° [18]. (There are other factors that degrade the reconstructed images such as a target's resonance phase variations over the wide UHF band, and commercial radio frequency interference.) Depending on what the angular orientation of the target is, one faces a complicated matching algorithm in the aspect angle domain to determine the target type and its orientation. The matching algorithm may also be unreliable and produce high false alarm rates or low detection rates (depending on the prescribed threshold level) due to the limited radar look angle data.

This problem can be circumvented by using circular SAR data collection of a spotlighted target region over 360° (full rotation). In reconnaissance with slant plane circular FOPEN/GPEN SAR, one is faced with comparable values for the altitude of the airborne radar (aircraft), the radius of the aircraft's circular path, and the radius of the spotlighted target area. Due to this fact, the classical approximation-based SAR imaging algorithms [5], [6] would fail in these UWB-UHF FOPEN/GPEN SAR imaging scenarios [10], [11].

One may utilize the reconstruction algorithm for the linear SAR system to perform inversion on circular SAR data with slant correction [12]. This may be achieved via the following steps.

i) Divide the 360° circular path into small arc segments (subapertures).
ii) Convert circular SAR data over the small arc centered at a given angle, e.g., θ into linear SAR data via motion compensation algorithms.
iii) Obtain the slant plane linear SAR reconstruction for the motion-compensated data.
iv) Interpolate from the slant plane into ground plane.
v) Rotate the ground plane reconstruction by −θ via interpolation.
vi) Repeat i)-v) for all arc segments θ values, and add the ground plane reconstruction results coherently.

There are two undesirable factors in this approach. First, it is computationally intensive [steps iv) and v) may be combined]. The other problem is that the numerous interpolation and motion-compensation errors may alter the subtle phase information in a target's UWB-UHF SAR signature, which is crucial for detecting man-made metallic structures (e.g., trucks and mines) from clutter (foliage and ground returns).

This paper presents a method for imaging from the slant plane circular SAR data. This approach is based on a Fourier analysis of the imaging system multidimensional impulse response, i.e., the slant plane Green's function. This analysis provides an understanding of information content of the circular SAR signal and methods to reconstruct the target function from it. In Section II, the circular SAR system model is developed, and its slant plane Green's function is identified. The Fourier properties of the slant plane Green's function are formulated in Section III. In Section IV, a method for inverting circular SAR signal, which exploits the
Fourier decomposition of the Green’s function, is presented. Resolution, reconstruction algorithm, and an example for this imaging system are provided in Section V. Three-dimensional (3-D) slant plane SAR imaging is discussed in Section VI.

II. SYSTEM MODEL

The imaging system geometry is shown in Fig. 1. The radar-carrying aircraft moves along a circular path with radius \( R \) on the plane \( z = z_0 \) with respect to the ground plane. Thus, the coordinates of the radar in the spatial domain as a function of the slow-time are \((x, y, z) = (R \cos \theta, R \sin \theta, z_0)\), where \( \theta \in [-\pi, \pi) \) represents the slow-time domain. As the radar moves along the circular synthetic aperture, its beam is spotlighted on the disk of radius \( X_0 \): denote it with \( D : X_0 \), centered at the origin of the spatial \((x, y)\) domain on the ground plane (the target region’s support). The effect of terrain’s altitude variations from the \( z = 0 \) plane is discussed in Section VI.

We denote the reflectivity function in the target region by \( f(x, y) \). We also define the slant range \( R_0 \equiv \sqrt{R^2 + z_0^2} \) and the following angles:

\[\theta_z \equiv \arctan \left( \frac{z_0}{R} \right) \quad \text{(slant angle)}\]

\[\pm \theta_x \equiv \arcsin \left( \frac{\pm X_0}{R} \right) \quad \text{(along path target angles)}.

From the side view of the imaging system geometry in Fig. 1, the target angles in the radial domain are within \( \arcsin(\pm X_0 \cos \theta_z)/R_0 \). For the following discussion, it is sufficient to define only \( (\theta_z, \theta_x) \). Moreover, in the formulation that follows, the spatial and spatial frequency amplitude functions are suppressed for notational simplicity.

We denote the transmitted UWB-UHF radar signal with \( p(t) \). The measured SAR signal in the slow-time and fast-time domains \((\theta, t)\) can be defined via the following:

\[s(\theta, t) \equiv \int \int f(x, y) d\theta d\omega.

The Fourier transform of the model in (1) with respect to the fast-time \( t \) is

\[s(\theta, \omega) \equiv P(\omega) \int \int f(x, y) g_0(x, y, \omega) \, dx \, dy \tag{2a}\]

where \( k = \omega/c \) is the wavenumber in the following:

\[g_0(x, y, \omega) \equiv \exp \left[ -j2k \sqrt{(x - R \cos \theta)^2 + (y - R \sin \theta)^2 + z_0^2} \right] \tag{2b}\]

for \((x, y) \in D : X_0\), and zero otherwise, is the SAR imaging system’s shift-varying impulse response (slant plane Green’s function) at the slow-time \( \theta \) and the fast-time frequency \( \omega \). For notational simplicity, the radar signal’s Fourier transform \( P(\omega) \), which appears on the right side of (2a), is not carried in the following discussion. (After fast-time matched filtering, this signal has the effect of an amplitude function.)

III. FOURIER PROPERTIES OF SLANT PLANE GREEN’S FUNCTION

To formulate a computationally manageable inversion for slant plane circular SAR, we first obtain an expression for the spatial Fourier transform of its Green’s function. For \( \theta = 0 \), the Green’s function in (2b) becomes

\[g_0(x, y, \omega) = \exp \left[ -j2k \sqrt{(x - R)^2 + y^2 + z_0^2} \right]. \tag{3}\]

This Green’s function is a shifted (the amount of shift is \( x = R \)) and space-limited (the support region is \( D : X_0 \)) version of the following free space Green’s function:

\[h(x, y, \omega) \equiv \exp \left( -j2k \sqrt{x^2 + y^2 + z_0^2} \right). \tag{4}\]
The two-dimensional (2-D) signal in (4) is a circularly symmetric function in the spatial \((x, y)\) domain. Thus, it can be expressed via the following:

\[
h(x, y, w) \equiv h_p(r)
\]

where

\[
h_p(r) \equiv \exp(-j2k\sqrt{r^2 + z_0^2})
\]

with \(r \equiv \sqrt{x^2 + y^2}\).

Using the Fourier properties of circular symmetric functions, one can show that the 2-D spatial Fourier transform of the signal \(h(x, y, w)\) is also a circularly symmetric function; i.e.,

\[
H(k_x, k_y, w) \equiv H_p(\rho)
\]

where \(H_0(\cdot)\) is the Hankel function of zero order in the following:

\[
H_p(\rho) \equiv \int r h_p(r) H_0(\rho r) \, dr = \int r \exp\left(-j2k\sqrt{r^2 + z_0^2}\right) H_0(\rho r) \, dr = \exp\left(-j\sqrt{4k^2 - \rho^2 z_0^2}\right)
\]

with \(\rho \equiv \sqrt{k_x^2 + k_y^2}\) [4].

If \(g_0(x, y, w)\) had been simply a shifted version of the free space Green’s function, i.e.,

\[
g_0(x, y, w) = h(x - R, y, w)
\]

then with the help of (8), we would have had the following for the spatial Fourier transform of the Green’s function at \(\theta = 0\):

\[
G_0(k_x, k_y, w) = H(k_x, k_y, w) \exp(-j k_x R) = H_p(\rho) \exp(-j k_x R) = \exp\left(-j\sqrt{4k^2 - \rho^2 z_0^2} - j k_x R\right).
\]

However, \(g_0(x, y, w)\) is a space-limited (windowed) version of \(h(x - R, y, w)\). The free space Green’s function \(h(\cdot)\) is a 2-D phase-modulated (PM) signal. Hence, using the windowing properties of PM signals [9], [13], one can show that this spatial window results in a spatial frequency window; i.e.,

\[
G_0(k_x, k_y, w) = W_0(k_x, k_y, w) \exp\left(-j\sqrt{4k^2 - \rho^2 z_0^2} - j k_x R\right)
\]

where \(W_0(\cdot)\) is an indicator function that represents the support of the Green’s function.

To find the window \(W_0(k_x, k_y, w)\), we should first determine the instantaneous frequency of the PM wave in the spatial domain, i.e., the partial derivatives of the phase function

\[
\gamma(x, y, w) \equiv 2k\sqrt{(x - R)^2 + y^2 + z_0^2}
\]

with respect to \(x\) and \(y\). These are

\[
K_x(x, y, w) = \frac{\partial \gamma}{\partial x} = \frac{2k(x - R)}{\sqrt{(x - R)^2 + y^2 + z_0^2}}
\]

\[
K_y(x, y, w) = \frac{\partial \gamma}{\partial y} = \frac{2k(y - R)}{\sqrt{(x - R)^2 + y^2 + z_0^2}}
\]

The spatial frequency support band of the Green’s function \(G_0(k_x, k_y, w)\) or \(W_0(k_x, k_y, w)\) is dictated by the set

\[
(k_x, k_y) \in \left\{ \left( \frac{\partial \gamma}{\partial x}, \frac{\partial \gamma}{\partial y} \right) : (x, y) \in \text{target area} \right\}.
\]

The target area is limited to the disk of radius \(X_0\), i.e., \(D : X_0\), in the spatial \((x, y)\) domain. In this case, with the help of (11a) and (11b), the support band of \(G_0(k_x, k_y, w)\) or \(W_0(k_x, k_y, w)\) in the polar spatial frequency domain \((\phi, \rho)\), where \(\phi \equiv \arctan(k_y/k_x)\), is approximately within the following region:

\[
|\rho - 2k\cos\theta_s| \leq 2k\sin^2\theta_s \sin\theta_s|\phi| \leq \theta_s.
\]

Thus, in the polar spatial frequency domain, i.e., \((\phi, \rho)\), the window function \(W_0\) is

\[
W_0(\phi, \rho, w) = \begin{cases} 1, & \text{for } |\rho - 2k\cos\theta_s| \leq 2k\sin^2\theta_s \sin\theta_s \\ 0, & \text{otherwise}. \end{cases}
\]

where

\[
W_1(\phi) = \begin{cases} 1, & \text{for } |\phi| \leq \theta_s \\ 0, & \text{otherwise}. \end{cases}
\]

and

\[
W_2(\rho, \omega) = \begin{cases} 1, & \text{for } |\rho - 2k\cos\theta_s| \leq 2k\sin^2\theta_s \sin\theta_s \\ 0, & \text{otherwise}. \end{cases}
\]

It can be shown that \(g_0(x, y, w)\) [see (2b)] is the \(\theta\)-rotated version of \(g_0(x, y, w)\) in the spatial \((x, y)\) domain. Thus, \(G_0(k_x, k_y, \omega)\) is also the \(\theta\)-rotated version of \(G_0(k_x, k_y, w)\) in the spatial frequency \((k_x, k_y)\) domain. Thus, we have

\[
G_0(\theta_k, k_y, \omega) = W_0(k_x, k_y, \omega) \cdot \exp\left[-j\sqrt{4k^2 - \rho^2 z_0^2} - j k_x R \cos\theta + k_y \sin\theta \right] = W_0(k_x, k_y, \omega) \cdot \exp\left[-j\sqrt{4k^2 - \rho^2 z_0^2} - j \rho R \cos(\theta - \phi) \right]
\]

where \(W_0(k_x, k_y, \omega)\) is the \(\theta\)-rotated version of \(W_0(k_x, k_y, w)\) in the spatial frequency \((k_x, k_y)\) domain. Thus, with the help of (14a)-(14c), in the polar spatial
frequency domain, i.e., $(\phi, \rho)$, the window function $W_\theta$ can be rewritten as follows:

$$W_{\theta p}(\phi, \rho, \omega) = \begin{cases} 1, & \text{for } |\rho - 2k \cos \theta_z| \leq 2k \sin^2 \theta_z, \\
0, & \text{otherwise.} \end{cases}$$

(16)

Note that $W_{\theta p}(\phi, \rho, \omega)$ is the $\theta$-shifted version of $W_{\theta p}(\phi, \rho, \omega)$ in the $\phi$ domain; i.e.,

$$W_{\theta p}(\phi, \rho, \omega) = W_{\theta p}(\phi - \theta, \rho, \omega) = W_1(\theta - \phi)W_2(\rho, \omega).$$

(17)

Fig. 2 shows examples of the Green’s function in the spatial frequency domain, i.e., $G_\theta(k_x, k_y, \omega)$ for various values of $\theta$ at the fast-time frequency of 300 MHz. For the ground plane scenario, i.e., $\theta_z = 0$ and $\cos \theta_z = 1$, the Green’s function spectrum is a pole distribution on the circle of radius $2k$ [4], [9]. The Green’s function possesses a very low energy spread around this circle, which corresponds to evanescent waves; the evanescent waves cannot be recorded by a realistic radar system where the range is significantly greater than the wavelength.

For the slant problem, i.e., $\theta_z \neq 0$ and $\cos \theta_z < 1$, the Green’s function possesses a spatial frequency (Doppler) spread around the circle of radius $\rho = 2k \cos \theta_z$ in the $(k_x, k_y)$ domain; the size of the spread [see (12)] is

$$\Delta_\rho = 2k \sin^2 \theta_z$$

$$\Delta_\phi = \pm \theta_z$$

which increases as $\theta_z$ or, equivalently, $X_0$ (the target area’s size) increases.

The physical meaning associated with the Doppler spread for the ground plane case, i.e., presence of evanescent waves, should not be associated with the spatial frequency spread for the slant case. The spread in this case is due to the fact that a nonzero slant makes the radar signal experienced by the ground targets have a wavelength other than $(2\pi c)/\omega$; this is a well-known fact in the classical radar theory [14], [15]. The effective wavenumber (wavelength) depends on the relative coordinates of the radar and the target, i.e., it varies with $(x, y, \theta)$. For a fixed $\theta$, the spread in the wavenumber domain is due to the variations of $(x, y)$, which yields the spread of $\Delta_\rho$ at $\rho = 2k \cos \theta_z$.

IV. INVERSION

Using the generalized Parseval’s theorem, the SAR system model in (2a) can be expressed via the following:

$$s(\theta, \omega) \equiv \int \int F(-k_x, -k_y)G_\theta(k_x, k_y, \omega) \, dk_x \, dk_y.$$  

(18)

For notational simplicity, we replace $F(-k_x, -k_y)$ with $F(k_x, k_y)$ in (18); this results in the rotation of the target function by $\pi$. The above double integral can be rewritten in the polar spatial frequency domain as shown in the following:

$$s(\theta, \omega) \equiv \int \int \rho F_p(\phi, \rho)G_{\theta p}(\phi, \rho, \omega) \, d\rho \, d\phi.$$  

(19)

Substituting (15)–(17) in (19) yields

$$s(\theta, \omega) = \int \int \rho F_p(\phi, \rho)W_1(\theta - \phi)W_2(\rho, \omega) \times \exp\left[-j \sqrt{4k^2 - \rho^2} z_0 - j\rho R \cos (\theta - \phi)\right] d\rho d\phi.$$  

(20)
Our objective is to recover $F_p$ from the 2-D inversion of the system model in (20).

The system model in (20) can be rewritten as follows:

$$s(\theta, \omega) = \int \Lambda(\rho, \omega) \Gamma(\theta, \rho) d\rho$$  \hspace{1cm} (21)

where

$$\Lambda(\rho, \omega) \equiv W_2(\rho, \omega) \exp \left( -j \sqrt{4k^2 - \rho^2 z_0} \right)$$  \hspace{1cm} (22)

and

$$\Gamma(\theta, \rho) \equiv \rho \int F_p(\phi, \rho)W_1(\theta - \phi) \exp \left[ -j \rho R \cos (\theta - \phi) \right] d\phi.$$  \hspace{1cm} (23)

Note that the signal $\Lambda(\rho, \omega)$ is a known 2-D signal; an example of this signal for a UWB-UHF radar signal is shown in Fig. 3(a).

Fig. 3(b) shows the magnitude of $\Lambda(\rho, \omega)$ at seven fast-time frequencies of the UWB-UHF radar signal. From (22), we can observe that this signal is a windowed PM signal in the $\rho$ domain; the PM signal, i.e., $\exp(-j \sqrt{4k^2 - \rho^2 z_0})$, and the
window, i.e., \( W_2(\rho, \omega) \), are both identified by the parameter \( k \) (or \( \omega \)). (The variation in the relative amplitude of these seven signals is due to the amplitude function \( 1/\sqrt{k} \) of the Green's function [4], which is suppressed in our formulation.) Thus, \( \Lambda(\rho, \omega) \) can be viewed as a set of nonorthogonal wavelets [17] in the \( \rho \) domain with parameter \( k \). Moreover, for a fixed \( \theta \), the operation in (21) can be viewed as passing the signal \( \Gamma(\theta, \rho) \) through a bank of these wavelets, the output of which is \( s(\theta, \omega) \). Recovering \( \Gamma(\theta, \rho) \) from this output can be viewed as performing the inverse wavelet transform [17]. This can be achieved via the following matrix operation.

Consider the system model in (21) for a fixed \( \theta \). Suppose the SAR signal is measured at \( N \) fast-time samples with sample spacing \( \Delta \). Thus, the sample spacing in the fast-time frequency domain \( \omega \) is \( \Delta \omega = 2\pi/N\Delta \) and the sample spacing in the wavenumber domain \( k \) is \( \Delta k = \Delta \omega/c \). Thus, the available discrete SAR data in the \( \omega \) domain are \( s(Q, w_n) \), where \( \omega_n = \omega_c + n\Delta \omega, -N/2 \leq n \leq N/2 - 1 \), with the wavenumber \( k_n = \omega_n/c \). We represent this database via the following \( N \times 1 \) vector:

\[
S_\theta = \left[ s(\theta, \omega_n); \frac{-N}{2} \leq n \leq \frac{N}{2} - 1 \right].
\] (24a)

We showed in (17c) that the radial spatial frequency window \( W_2(\rho, \omega) \) is centered around \( \rho = 2k \cos \theta \). Thus, the discrete values of \( \rho \) that should be considered for the model in (21) are \( \rho_n = 2k \cos \theta_n \), for \( -N/2 \leq n \leq N/2 - 1 \). We define the following known \( N \times N \) matrix (discrete \( \Lambda \); system kernel) as

\[
\Lambda = \left[ \Lambda(\rho_m, \omega_n); \frac{-N}{2} \leq m, n \leq \frac{N}{2} - 1 \right].
\] (24b)

For the unknown, i.e., the \( \Gamma \) signal, we define the following \( N \times 1 \) vector:

\[
\Gamma_\theta = \left[ \Gamma(\theta, \rho_m); \frac{-N}{2} \leq m \leq \frac{N}{2} - 1 \right].
\] (24c)

Using (24a)-(24c), the system model in (21) can be expressed via the following linear model:

\[
S_\theta = \Lambda \Gamma_\theta.
\] (25)

From (25), we have the following inversion:

\[
\Gamma_\theta = \Lambda^{-1} S_\theta.
\] (26)

As we mentioned earlier, the wavelets \( \Lambda(\rho, \omega) \) are nonorthogonal in the \( \rho \) domain. However, it can be shown that these wavelets are approximately orthogonal (see Appendix A). In this case, the system kernel's inverse matrix can be approximated via its conjugate transpose; i.e.,

\[
\Lambda^{-1} \approx \Lambda^*.
\] (27)

In this case, the inversion in (26) becomes

\[
\Gamma(\theta, \rho_m) = \sum_{\omega_n} s(\theta, \omega_n) \Lambda^*(\rho_m, \omega_n).
\] (28a)

The continuous form of the inversion (28a) is

\[
\Gamma(\theta, \rho) \equiv \int_{\omega} s(\theta, \omega) \Lambda^*(\rho, \omega) d\omega
\] (28b)

which is evaluated at \( \rho = \rho_m, -N/2 \leq m \leq N/2 - 1 \). This is demonstrated in Appendix B. The mathematical operation in (28b) is based on the well-known principle of wavefront reconstruction, which is also utilized in other wave equation-based coherent imaging problems [18].

The outcome of the inversion in (26) or (28a) and (28b) are the samples of the signal \( \Gamma(\theta, \rho) \) at the discrete values of \( \rho = \rho_m = 2k \cos \theta_n, -N/2 \leq m \leq N/2 - 1 \); i.e.,

\[
\Gamma(\theta, 2k \cos \theta_n) = 2k \cos \theta_n \int P(\phi, 2k \cos \theta_n) W_j(\theta - \phi) \cdot \exp \left[-j2k \cos \theta_n R \cos (\theta - \phi)\right] d\phi.
\] (29)
Dividing both sides of (29) with $2k \cos \theta_z$, one obtains
\[ s_1(\theta, 2k \cos \theta_z) = \int F_p(\phi, 2k \cos \theta_z)W_1(\theta - \phi) \cdot \exp \left[ -j2k \cos \theta_z R \cos (\theta - \phi) \right] d\phi \] (30)

where
\[ s_1(\theta, 2k \cos \theta_z) = \frac{1}{2k \cos \theta_z} \Gamma(\theta, 2k \cos \theta_z) \] (31)
is a known signal (processed measured data). Thus, our next task is to recover

The system model in (30) can be expressed via the following convolution in the angle $\theta$ domain:
\[ s_1(\theta, 2k \cos \theta_z) = F_p(\theta, 2k \cos \theta_z) \ast [W_1(\theta) \exp (-j2kR \cos \theta_z \cos \theta)] \] (32)
where $\ast$ denotes convolution in the $\theta$ domain. A similar system model and its deconvolution are also encountered in ground plane circular SAR where $\theta_z = 0$ [7, 9]. To solve for the target function $F_p(\theta, 2k \cos \theta_z)$ from (32), we have to deconvolve the known signal
\[ W_1(\theta) \exp (-j2kR \cos \theta_z \cos \theta) \]
from $s_1(\theta, \omega)$. The deconvolved signal, call it $s_2(\theta, \omega)$, in the frequency domain of $\theta$ is obtained via
\[ F(\theta)[s_2(\theta, \omega)] = \frac{\mathcal{F}(\theta)[s_1(\theta, \omega)]}{\mathcal{F}(\theta)[W_1(\theta) \exp (-j2kR \cos \theta_z \cos \theta)]}. \] (33)
The deconvolution kernel in (33), i.e.,
\[ W_1(\theta) \exp (-j2kR \cos \theta_z \cos \theta), \]
is an amplitude modulated–phase modulated (AM–PM) signal. The PM component has the one-dimensional (1-D) Fourier transform with respect to the slow-time $\theta$ [4, 7, 9], as follows:
\[ \mathcal{F}(\theta)[\exp (-j2kR \cos \theta_z \cos \theta)] = H^{(2)}(2kR \cos \theta_z) \exp \left( -j\frac{\pi\xi}{2} \right) \] (34)
where $H^{(2)}$ is the Hankel function of the second kind, $\xi$ order. Using the windowing properties of PM signals [9, 13, 18] and the fact that the support of the window $W_1(\theta)$ is $|\theta| \leq \theta_x$ [see (14b)], the Fourier transform of the deconvolution kernel is the right side of (34) for
\[ |\xi| \leq 2kR \cos \theta_z \sin \theta_x = 2kX_0 \cos \theta_x \] (35)
and zero otherwise.

Thus, the deconvolution in (33) is implemented via the following (matched filtered) form:
\[ F(\theta)[s_2(\theta, \omega)] = \mathcal{F}(\theta)[s_1(\theta, \omega)]H^{(1)}(2kR \cos \theta_z) \cdot \exp \left( j\frac{\pi\xi}{2} \right) \] (36)
where $H^{(1)}$ is the Hankel function of the first kind, $\xi$ order. It has been suggested that the kernel on the right side of (34) can be approximated via $\exp (\sqrt{2k^2 \cos^2 \theta_z - \xi^2})$ to avoid evaluating Hankel functions and, thus, reducing the computational load [9]. For the SAR reconnaissance problems, where $R, z_0$, and $X_0$ are comparable, this approximation is not accurate and should not be used.

From the system model in (32), the resultant deconvolved signal is the target function’s polar signal in the spatial frequency domain, i.e.,
\[ F_p(\theta, 2k \cos \theta_z) = s_2(\theta, \omega). \] (37)
The polar samples of $F_p(\theta, 2k \cos \theta_z)$ are then converted to the samples of $F(k_x, k_y)$ where
\[ k_x(\theta, \omega) = 2k \cos \theta_z \cos \theta \]
\[ k_y(\theta, \omega) = 2k \cos \theta_z \sin \theta \] (38)
via an interpolation algorithm. The 2-D inverse Fourier transform of this signal is the desired image. This completes the multidimensional signal theory for the inversion in slant plane circular SAR.

We should point out that one can also utilize the Fourier decomposition of the slant plane Green’s function to develop a direct slant plane inversion for linear stripmap or spotlight SAR. However, the classical approach of using the ground plane reconstruction on the slant plane linear SAR data and then using a single interpolation to transform the slant plane image into ground plane is not computationally intensive as compared with the direct method.

V. RECONSTRUCTION ALGORITHM

The sampling constraints and signal processing issues associated with slant plane round SAR are identical to those of ground plane circular SAR [7, 9] with $k$ replaced with $k \cos \theta_z$. Based on the inverse theory presented in Section IV, the following steps can be used to reconstruct the target function from the measured slant plane circular SAR signal.

1) Compute the 1-D Fourier transform of the SAR signal with respect to the fast-time to obtain $s(\theta, \omega)$.
2) Compute the samples of the signal $\Lambda(\rho, \omega)$ from (22) at the sample points $(\rho_m, \omega_n) = (2k_m \cos \theta_z, \omega_n)$ to form the matrix $\Lambda$ in (24b). Compute the inverse of this matrix to obtain $\Lambda^{-1}$. We use (27), i.e., $\Lambda^{-1} \approx \Lambda^T$, for this purpose.
3) For a fixed radar (aircraft) look angle $\theta$ (i.e., slow-time), form the vector $\Sigma_\theta$. Compute the samples of the $\Gamma$ signal, which is defined in (29), i.e., the vector $\Gamma_\theta$, from the matrix operation in (26).
4) Compute the samples of the signal $s_1(\theta, 2k \cos \theta_z)$ from the knowledge of $\Gamma(\theta, 2k \cos \theta_z)$ using (31).
5) Compute the 1-D Fourier transform of $s_1(\theta, 2k \cos \theta_z)$ with respect to $\theta$. Use this signal to perform the deconvolution in (36). Obtain the inverse Fourier transform of the resultant; this yields the signal $s_2(\theta, \omega)$, which is equal to the polar samples of the target function’s spatial Fourier transform, i.e., $F_p(\theta, 2k \cos \theta_z)$ [see (37)].
6) Use polar to rectangular interpolation to reconstruct $F(k_x, k_y)$ from the polar data $F_p(\theta, 2k \cos \theta_z)$ [see (38)]. The inverse spatial Fourier transform of the resultant is the target function $f(x, y)$ in the spatial domain.

The resolution of a reflector in this imaging system depends on the interval of radar aspect angle over which it is observable, i.e., the support of its SAR signature in the $\theta$ domain. For instance, suppose the $i$th target’s SAR signature support is approximately within $[\psi_i - \psi_0, \psi_i + \psi_0]$ in the $\theta$ domain, where $\psi_i$ is a quantity that depends on the target’s angular orientation in the spatial domain, and $\psi_0 < \pi/4$ depends on the physical shape or type of the target [18]. (Most man-made targets, such as trucks and tanks, possess this type of SAR signature.) Then, the target’s resolution in the $\psi_i$-rotated version of the spatial domain, call it the $(x_i, y_i)$ domain, can be found via the classical SAR resolution [9] as follows:

$$
\Delta x_i = \frac{\pi}{\rho_{\text{max}} \sin \psi_0}
$$

$$
\Delta y_i = \frac{\pi}{2k_z \cos \theta_z \sin \psi_0}
$$

where

$$
\rho_{\text{max}} = 2k_{\text{max}} \cos \theta_z
$$

$$
\rho_{\text{min}} = 2k_{\text{min}} \cos \theta_z.
$$

Fig. 4. (a) Target model. (b) Gradient angle. (c) Target edge model. (d) SAR reconstruction with full rotation data, $|\theta| \leq \pi$. 
The spatial point spread function (PSF) of the targets with \( \psi_0 > \pi/4 \) are difficult to quantify. For the class of omnidirectional reflectors, we have \( \psi_0 = \pi \); i.e., these targets are observable to the radar at all slow times (aspect angles). (Tree trunks and buried mines are approximately omnidirectional reflectors.) The support band of an omnidirectional target located at \((x, y) = (0, 0)\) in the spatial frequency domain is

\[
\rho_{\text{min}} \leq \sqrt{k_x^2 + k_y^2} \leq \rho_{\text{max}}.
\]

Thus, the PSF of this target in the spatial domain is the difference of two Airy patterns [9], [16] as in

\[
\text{PSF}(x, y) = \rho_{\text{max}} \frac{J_1(r \rho_{\text{max}})}{r} - \rho_{\text{min}} \frac{J_1(r \rho_{\text{min}})}{r}
\]

where \( r = \sqrt{x^2 + y^2} \), and \( J_1 \) is the Bessel function of the first kind, first order. In the extreme case of \( \rho_{\text{min}} = 0 \), the radial resolution becomes

\[
\Delta_0 \equiv \frac{\pi}{\rho_{\text{max}}}
\]

which is never achieved in practice.

For typical values of \((\rho_{\text{max}}, \rho_{\text{min}})\) in UWB-UHF SAR systems, the PSF in (41) has a peak at \( r = 0 \), its first zero-crossing is around \( r = \pm \Delta_0 \), and it has a damping behavior that vanishes around \( r = \pm \pi/\rho_{\text{max}} \approx \rho_{\text{min}} \) [i.e., the range resolution in (39)]. In this case, the radial resolution is approximately within \( a_0 \Delta_0 \) where \( 1 \leq a_0 \leq 2 \).

Consider a slant plane circular SAR scenario with the following parameters: \( R = 500 \text{ m} \), \( z_0 = 500 \text{ m} \), radar (UHF) carrier frequency \( f_c = 300 \text{ MHz} \); and radar (UWB) baseband bandwidth \( \pm f_0 = \pm 125 \text{ MHz} \). The ground target is an aircraft that is shown in Fig. 4(a). This target is centered at \((x, y) = (50, 50) \) in the spatial domain. To simulate the SAR signature of this target, the following steps are used. We first obtain the angle of the gradient of the aircraft’s image; this gradient angle, denote it with \( \pi/2 + \Psi(x, y) \), is shown in Fig. 4(b). We then use an edge operator and thresholding to obtain the target edge model of Fig. 4(c). This edge model is composed of 334 reflectors, the coordinates of which are denoted with \( x_i, y_i, i = 1, \cdots, 334 \).

The relative gradient angle with respect to the target/radar coordinates at these edge points, i.e.,

\[
\alpha_i(\theta) \equiv \Psi(x_i, y_i) - \arctan \left( \frac{y_i - R \sin \theta}{x_i - R \cos \theta} \right)
\]

represents the slope of the aircraft’s boundary as seen by the radar at the aspect angle \( \theta \). When this angle is zero, the \( i \)th reflector appears to be at broadside with respect to the radar. This angle is the measure used to determine the power of the echoed radar signal at a given reflector. For the \( i \)th reflector, we model the amplitude of its SAR signature at the fast-time frequency \( \omega \) and the aspect angle \( \theta \) via the following Hanning window:

\[
A_i(\theta) \equiv \begin{cases} 
0.5 + 0.5 \cos [2\alpha_i(\theta)], & \text{for } |\alpha_i(\theta)| < \frac{\pi}{2}; \\
0, & \text{otherwise}.
\end{cases}
\]

For this amplitude pattern, the support band of the target’s SAR signature is approximately \( \pm \psi_0 \approx \pm \pi/4 \). (Other types of windows or parameters may be used that would yield different \( \psi_0 \) values.)

The measured SAR signal is computed via the following:

\[
s(\theta, \omega) = \sum_{i=1}^{334} A_i(\theta) g_\theta(x_i, y_i, \omega).
\]

Fig. 5(a) shows the resultant SAR signal, i.e., \( s(\theta, \omega) \). Fig. 5(b) is the full rotation, i.e., \( |\theta| \leq \pi \), SAR reconstruction.
of the target in the spatial frequency domain $F(k_x, k_y)$. The full rotation SAR reconstruction in the spatial domain is shown in Fig. 4(d). Fig. 6(a)--(d) shows the spatial domain reconstruction of the target when a $\pm \pi/4$ angular interval of the SAR data which is centered around, respectively, $\theta = 0, \pi/2, \pi, -\pi/2$, is used in the processing.

The plane wave approximation-based inversion for slant plane circular SAR [10], [11] can be shown to be based on the following:

$$F_p(\theta, 2k \cos \theta) \approx s(\theta, \omega) g_0^2(0, 0, \omega)$$

$$= s(\theta, \omega) \exp \left(j2k \sqrt{R^2 + z_0^2} \right). \quad (43)$$

The full rotation reconstruction of the target using this method is shown in Fig. 7. The method can be shown to fail if the constraint

$$\frac{k \cos \theta X_0^2}{R_0} \ll 1$$

is not satisfied [9].

VI. 3-D IMAGING

The system model and inversion, which were formulated in the previous sections, used a target model that was located on the plane $z = 0$. Thus, the reconstructed image can be viewed
Next, we consider the case of slant plane circular SAR. For the 3-D target scene, the system model in (1) should be modified as in (44), shown at the bottom of the page. Suppose there exists only a single reflector at \((x, y, z)\) with altitude \(z(x, y)\) in the target scene. The distance of this target from the radar at the slow-time \(\theta\) is

\[
\sqrt{(x - R \cos \theta)^2 + (y - R \sin \theta)^2 + (z(x, y) - z_0)^2}.
\]

Thus, the target’s slant-range, which is

\[
\sqrt{(x - R \cos \theta)^2 + (z(x, y) - z_0)^2}
\]

varies with the slow-time. Hence, this slow-time dependent slant-range should be incorporated in any coherent processing in the slow-time domain. The inversion in Section IV provides such a coherent processing which is sensitive to the target’s altitude.

To identify and quantify the sensitivity of the algorithm in the altitude domain (i.e., the altitude resolution), consider the case of \(|z(x, y)| \ll R_0\). For this, the target’s distance from the radar can be approximated via

\[
\sqrt{(x - R \cos \theta)^2 + (y - R \sin \theta)^2 + (z(x, y) - z_0)^2} \approx 
\sqrt{(x - R \cos \theta)^2 + (y - R \sin \theta)^2 + z_0^2 - \sin \theta z(x, y)}.
\]

(45)

Using (45) in the system model (44), and following the steps which led to the inversion in (37), we obtain

\[
F_p(\theta, 2k \cos \theta_z) \exp[j2k \sin \theta_z z(x, y)] = s_2(\theta, \omega).
\]

(46)

Equation (46) indicates that there exists a PM term, \(\exp[j2k \sin \theta_z z(x, y)]\), that is not incorporated in the inversion if the target scene is assumed to lie on the plane \(z = 0\). This PM term is modulating the target function in the frequency domain. This results in a target reconstruction that is smeared and/or shifted in the spatial \((x, y)\) domain. The effect depends on the target’s support region in the frequency domain.

For instance, if the target is an omnidirectional located at \((x, y) = (0, 0)\) (i.e., \(F(k_x, k_y) = 1\)), its signature in the spatial frequency domain is

\[
\exp[j2k \sin \theta_z z(x, y)] = \exp[j \sqrt{k_x^2 + k_y^2} \tan \theta_z z(x, y)]
\]

(47)

for \(\rho_{\min} \leq \sqrt{k_x^2 + k_y^2} \leq \rho_{\max}\), where \((k_x, k_y)\) are defined in (38). For \(z(x, y) = 0\), the resultant PSF was shown in (41). For a general \(z(x, y)\), the PSF is approximately a donut-shaped structure with the mean radius of \(\tan \theta_z z(x, y)\) and the width of approximately \(\pm a_0 \Delta \theta_0\) [1 \(\leq a_0 \leq 2\); see (42)].
The peak power of the PSF for \( z(x, y) \neq 0 \) can be shown to be lower than the peak power of the PSF with \( z(x, y) = 0 \) by

\[
10 \log \left[ \frac{a_0 \Delta_0}{4 \tan \theta_z |z(x, y)|} \right] \text{dB} \tag{48}
\]

if \( a_0 \Delta_0 < \tan \theta_z |z(x, y)| \), i.e., the radius of the donut is greater than half of its width. (The power ratio for the case of \( a_0 \Delta_0 \geq \tan \theta_z |z(x, y)| \) has a lengthy expression that is not shown here.) When the support of the target’s SAR signature is approximately within \([\psi_1 - \psi_0, \psi_1 + \psi_0]\) in the \( \theta \) domain, with \( \psi_0 < \pi/4 \), the reconstruction of this target appears smeared and shifted by \( \tan \theta_z z(x, y) \) in the \( x_1 \) domain, where \((x_i, y_i)\) is the \( \psi_i \) rotated version of the spatial \((x, y)\) domain. The power loss due to smearing is dictated by \( (48) \). (The plane wave approximation-based inversion exhibits the same effect due to the terrain’s altitude variations in addition to its system model errors.)

Fig. 8(a)–(c) exhibits this phenomenon for the SAR system, which was cited earlier. For this case, we use 11 reflectors on the line \( y = 0 \), which are located at \( x = 0, \pm 4, \pm 8, \pm 12, \pm 16, \pm 20 \) m, with respective altitudes of \( z = 0, \pm 0.5, \pm 1, \pm 1.5, \pm 2 \) m. The targets located at \( x \geq 0 \) are omnidirectional; the targets located at \( x < 0 \) have the Hanning amplitude pattern. The results indicate that, as the target altitudes deviates from the reconstruction plane of \( z = 0 \), its signature gets weaker (more smeared); i.e., the targets on the \( z = 0 \) plane appeared focused while targets at other altitudes appear out of focus.

The above results indicate that one can select a set of \( z \) values, e.g., \( z_n = z_0 + n \Delta_z \) (\( n = 0, \pm 1, \pm 2, \cdots \)), to form a 3-D image of the target scene, i.e., \( f(x, y, z_n) \), for \( z_n = z_0 + n \Delta_z \) (\( n = 0, \pm 1, \pm 2, \cdots \)). This can be achieved in two ways. One approach is to construct the system kernel \( \Lambda \) in (22) at each one of the desired altitude values \( z_n \), i.e.,

\[
\Lambda(\rho, \omega) \equiv W_2(\rho, \omega) \exp \left( -j \sqrt{4k^2 - \rho^2} z_n \right)
\]

and follow the steps that led to the inversion in (37) to form \( F(\kappa_x, \kappa_y, z_n) \) and its inverse spatial Fourier transform \( f(x, y, z_n) \); this has to be repeated for all the desired \( z_n \) values.

The other approach is an approximation that works in most practical scenarios where \( |z_n - z_0| \ll z_0 \) (in this case, \( W_2 \) does not change significantly with the variations of \( z_n \), and the phase error introduced by the approximation in (45) is small. For this method, we first reconstruct the spatial frequency function \( F(\kappa_x, \kappa_y, z_0) \) via the reconstruction steps that were outlined in Section V. Then, using (46) and (47), the spatial frequency reconstruction for the altitude \( z_n \) can be formed via

\[
F(\kappa_x, \kappa_y, z_n) = F(\kappa_x, \kappa_y, z_0) \exp \left( -j \sqrt{k_x^2 + k_y^2} \tan \theta_z (z_n - z_0) \right) \tag{49}
\]

where \( \theta_z = \arctan \left( z_0/R \right) \) is the slant angle for the mid-altitude \( z_0 \). The 2-D inverse spatial Fourier transform of \( F(\kappa_x, \kappa_y, z_n) \) in (49) is the desired image \( f(x, y, z_n) \). The selection of \( \Delta_z \) (altitude resolution) depends on the power measure (48) that represents out-of-focusing effects of the targets that fall out of the reconstruction plane. The focusing property of the slant plane circular SAR is unique. As we mentioned earlier, the same property is not true for the slant plane linear SAR.

One of the applications of this 3-D imaging is in GPEN SAR for detection of mines and underground tunnels. In GPEN SAR problems, the speed of wave propagation in the soil, where the target (i.e., a mine or underground tunnel) is buried, is different from that of the air. Suppose the radar-carrying aircraft’s altitude from ground level is \( z_G \), and we are interested in imaging targets that are buried at \( z_T \) below ground level. If the average index of refraction of the soil is \( n \), then the effective altitude of these targets (which are located at the distance \( z_T \) from the ground level) with respect to the radar is approximately equal to

\[
z_0 \equiv z_G + \left( 1 + \frac{n-1}{\sin \theta_z} \right) z_T. \tag{50}
\]

For imaging these targets, the value of \( z_0 \) in (50) should be used to construct the system kernel \( \Lambda \) in (22) and the inversion (inverse wavelet transform) in (28b). In this case, (48), as a measure of the relative power of the reconstruction signature of the targets that are located at \( z_0 + z \) in the soil, becomes

\[
10 \log \left[ \frac{a_0 \Delta_0}{4 \tan \theta_z |z| + \frac{4}{\cos \theta_z} |(n-1)|z|} \right] \text{dB}. \tag{51}
\]

Equation (51) indicates improved altitude resolution when \( n > 1 \) which is the case for the soil’s index of refraction. To quantify the power of the reconstruction signature of the ground targets on the imaging plane of the altitude \( z_0 \) in (50), \( z = z_T \) should be used in (51).

VII. CONCLUSIONS

In this paper, we developed an inversion for the slant plane SAR data that are collected over a circular flight path. We demonstrated the utility of the method in reconnaissance with UWB-UHF FOPEN/GPEN SAR using a simulated target, an aircraft. The strength of the echoed signal from a reflector on the target’s surface was modeled as a function of the surface gradient angle and the radar’s aspect angle. This model helped us to demonstrate that the reconstruction made from a partial \((\pm 45^\circ)\) segment of the circular synthetic aperture provides limited shape information (or SAR coherent signature), which may be used for target detection and identification purposes. Meanwhile, the full rotation reconstruction clearly showed the target’s boundary structure. The utility of the slant plane circular SAR in 3-D imaging was discussed.

Finally, similar to the other SAR systems, slant plane circular SAR system is susceptible to the random (nonlinear) motions of the radar-carrying aircraft from its prescribed circular path. To compensate for these motion errors, the aircraft should be equipped with a global positioning system (GPS). The GPS data of the aircraft, which carry information regarding the position of the aircraft at various
bursts of the radar along its flight path (i.e., slow time), can be used to compensate for the nonlinear motion [9].

APPENDIX A

The inner product of the wavelets at, e.g., $\omega_m$ and $\omega_n$, is

$$\alpha_{mn} \equiv \int \Lambda(\rho, \omega_m) \Lambda^*(\rho, \omega_n) \, d\rho. \quad (A1)$$

Using (22), (A1) can be rewritten as follows:

$$\alpha_{mn} = \int W_2(\rho, \omega_m) \exp \left( -j \sqrt{4k_m^2 - \rho^2} z_0 \right)$$

$$\cdot W_2(\rho, \omega_n) \exp \left( j \sqrt{4k_n^2 - \rho^2} z_0 \right) \, d\rho$$

$$= \int W_2(\rho, \omega_m) W_2(\rho, \omega_n)$$

$$\cdot \exp \left[ j \left( \sqrt{4k_n^2 - \rho^2} - \sqrt{4k_m^2 - \rho^2} \right) z_0 \right] \, d\rho. \quad (A2)$$

The two PM signals $\exp(-j \sqrt{4k_m^2 - \rho^2} z_0)$ and $\exp(j \sqrt{4k_n^2 - \rho^2} z_0)$ on the right side of (A2) are highly fluctuating signals in the support region of $W_2(\cdot)$. If $k_m \neq k_n$, the instantaneous frequency of these two PM waves never match over the integration interval. Thus, if the integral is
approximated by the method of stationary phase, then the result is approximately equal to zero when \( k_m \neq k_n \), and a nonzero constant (e.g., one) otherwise. This can also be verified numerically for the UWB-UHF radar signal that was cited in the paper.

**APPENDIX B**

We want to prove the validity of the inversion in (28b); i.e.,

\[
\Gamma(\theta, \rho) \equiv \int s(\theta, \omega) A^*(\rho, \omega) \, d\omega. \tag{B1}
\]

Substituting for \( A \) from (22) in (B1) and after some rearrangements, one obtains the following:

\[
\Gamma(\theta, \rho) = \int_\gamma \Gamma(\theta, \gamma) I(\rho, \gamma) \, d\gamma \tag{B2}
\]

where

\[
I(\rho, \gamma) \equiv \int_\omega A(\gamma, \omega) A^*(\rho, \omega) \, d\omega \\
= \int_\omega W_2(\gamma, \omega) W_2(\rho, \omega) \\
\cdot \exp \left[ j(\sqrt{4k^2 - \rho^2} - \sqrt{4k^2 - \gamma^2})z_0 \right] \, d\omega. \tag{B3}
\]

The two PM signals \( \exp(-j\sqrt{4k^2 - \gamma^2}z_0) \) and \( \exp(j\sqrt{4k^2 - \rho^2}z_0) \) on the right side of (B3) are highly fluctuating signals in the support region of \( W_2(\cdot) \). If \( \gamma \neq \rho \), the instantaneous frequency of these two PM waves never match over the integration interval. Thus, if the integral is approximated by the method of stationary phase, then the result is approximately equal to zero when \( \gamma \neq \rho \), and a nonzero constant (e.g., one) otherwise. Using this in (B2), one can show the validity of (B1).

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